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Condensational theory of stationary tornadoes

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ABSTRACT

Using the Bernoulli integral for air streamline with condensing water vapor a stationary axisymmetric tornado circulation is described. The obtained profiles of vertical, radial and tangential velocities are in agreement with observations for the Mulhall tornado, world's largest on record and longest-lived among the three tornadoes for which 3D velocity data are available. Maximum possible vortex velocities are estimated.

1. The condensational pressure potential

Tornado circulation induced by water vapor condensation can be described as follows. Condensation of water vapor in the adiabatically ascending air results in a drop of air pressure by $\Delta p = p_v$, where p_v is water vapor pressure at the Earth's surface. The decrease of pressure along the vertical axis sustains the ascending air motion with vertical velocity w and induces a compensating horizontal air inflow with radial velocity u. The converging radial flow has maximal velocity at the surface, where the magnitude of the condensation-induced pressure drop is the largest. Radial velocity approaches zero at a certain height $z = h$, which approximately coincides with the cloud height. In the upper atmosphere at $z > h$ the condensed water is transported away from the condensation area by the strong updraft and outgoing air flow. It precipitates at a considerable distance from the center of the condensation area.

The continuity equation in the cylindrical system of coordinates relates radial u and vertical w velocities of the axially symmetrical vortex as $w = (h/r)(\partial ur/\partial r)$. The vertical and horizontal pressure gradient forces induced by condensation are $\Delta p/h$ and $\partial p/\partial r$, respectively. Equating the power of the vertical and radial air flow, $u\,\partial p/\partial r = w(\Delta p/h)$, and accounting for the continuity equation, we obtain $\partial p/\partial r = \Delta p(ur)^{-1}(\partial ur/\partial r)$. This corresponds to pressure potential $p = \Delta p \ln ur + \text{const}$ [1]. Its exact derivation is given in work [2].

2. The Bernoulli integral for condensation-induced tornadoes

For the high wind velocities of intense vortices to arise, the condensational pressure gradients within both tornadoes and hurricanes must significantly exceed turbulent friction. In such a case, the Euler equations possess a Bernoulli integral for the streamline:

$$B(r) \equiv \frac{1}{2}\rho\left(u^2 + w^2 + v^2\right) + \Delta p \ln ur = B(r_1), \tag{1}$$

$$p(r) = p_1 + \Delta p \ln ur, \qquad \Delta p = p_v \equiv \rho\frac{u_c^2}{2}, \tag{2}$$

$$p_1 \equiv p(r_1), \qquad w = \frac{h}{r}\frac{\partial ur}{\partial r}, \qquad v = \frac{a}{r}. \tag{3}$$

Here u, w and v are the radial, vertical and tangential velocities, respectively, r is distance from the center of the condensation area, $r = r_1$ is the outer border of the condensation area, ρ is air density, u_c is the velocity scale determined by water vapor condensation, a is angular momentum per unit air mass, and $z < h$ is the region of converging streamlines ($u > 0$).

It is convenient to use the following units

$$\Delta p = 1, \qquad u_c = 1, \qquad \rho = 2, \qquad r_1 = 1. \tag{4}$$

In these units, the Bernoulli integral and the pressure potential become dimensionless

$$B(r) - B(r_1) = u^2 - u_1^2 + w^2 - w_1^2 + \left(\frac{a^2}{r^2} - a^2 + \ln\frac{ur}{u_1}\right) = 0, \tag{5}$$

$$p(r) = p_1 + \ln\frac{ur}{u_1}, \tag{6}$$

where $u_1 \equiv u(r_1)$, $w_1 \equiv w(r_1)$, and $a \equiv v_1 \equiv v(r_1)$.

Let us introduce a new variable $y \equiv ur/u_1$. Then Eq. (5) takes the form of a nonlinear differential equation on y:

$$y' = \frac{r}{u_1 h}\sqrt{u_1^2\left(h^2 {y'_1}^2 + 1 - \frac{y^2}{r^2}\right) - \left(\frac{a^2}{r^2} - a^2 + \ln y\right)}, \tag{7}$$

$$w \equiv u_1\frac{h}{r}y', \qquad u \equiv u_1\frac{y}{r}, \qquad v = \frac{a}{r}, \qquad p = p_1 + \ln y, \tag{8}$$

$$y \equiv \frac{ur}{u_1}, \qquad y' \equiv \frac{dy}{dr}, \qquad y'_1 \equiv y'(r_1). \tag{9}$$

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Real solution of Eq. (7) exists at those r only, where the expression under the square root in Eq. (7) is positive. The internal radius $r = r_0$, where condensation ceases, is obtained by equating the last term in the round brackets in Eqs. (5) and (7) to zero at $y(r_0) = r_0$, which is equivalent to $u(r_0) = u_1$. Condensation commences at r_1 and ceases at r_0 at one and the same radial velocity u_1. As follows from Eq. (7), the following relationships are simultaneously satisfied:

$$\frac{a^2}{r_0^2} - a^2 + \ln r_0 = 0, \qquad y_0 \equiv y(r_0) = r_0, \tag{10}$$

$$y_1 \equiv y(r_1) = 1, \qquad y_0' \equiv y'(r_0) = r_0 y_1', \tag{11}$$

$$u_0 \equiv u(r_0) = u_1, \qquad w_0 \equiv w(r_0) = w_1 = u_1 h y_1'. \tag{12}$$

At $r = r_0$ all condensational potential energy is converted to the kinetic energy of rotation. At $r < r_0$, real solutions of Eq. (7) for velocities u, v, and w do not exist.

Eq. (7) is a first-order differential equation with one boundary condition: $y_1 = 1$ ($u(r_1) = u_1$) or $y_0 = r_0$ ($u(r_0) = u_1$). If at fixed y_1 one considers the constant y_1' in Eq. (7) as a free parameter, then in the general case the interval, where real solutions exist, does not include the point $r = r_0$, which means that the maximum velocity $v_{max} \sim a/r_0$ is not reached and the tornado does not exist.

Tornado exists, when the interval of real solutions comprises the point r_0 defined by Eq. (10). Solution of Eq. (7), that is real within the range $r_0 \leqslant r \leqslant 1$, is obtained by setting the boundary condition on y at r_0 as $y_0 = r_0$ and choosing y_1' at given u_1, a and h such that $y_1 = 1$.

3. Comparison with observations

Data of three-dimensional circulation (the dependencies of the velocities u, w and v on distance r from the tornado center) have only recently become available and exist for three tornadoes [3–5]. We shall consider the Mulhall tornado (Oklahoma, 3 May 1999), which is the longest-lived (1 hour 20 min [6]) and longest-observed (18 min [5]) among the three as well as world's largest tornado on record [5].

According to empirical observations, the intense tornadoes can occur, when the mean relative humidity at $z \lesssim 1$ km is not lower than 75–85% [7]. At a characteristic surface temperature 30°C [8] and 80% relative humidity the vapor pressure at the surface is $p_v = \Delta p \simeq 30$ hPa. Taking air density $\rho = 1.15\ \mathrm{kg\,m^{-3}}$ in Eq. (2), we obtain the characteristic velocity $u_c = 73\ \mathrm{m\,s^{-1}}$. Velocities v_1 and u_1 at the external border $r = r_1$ must be the functions of translational velocity U (speed of movement of tornado as a whole). We put radial velocity $u_1 = U/\pi$ [2], taking into account that the flux of moist air via tornado cross-section $2r_1U$ is equal to the flux via tornado circumference $2\pi r_1 u_1$. We put tangential velocity $v_1 = 2U/\pi$ assuming that the angular momentum of the main streamline that delivers moist air into the condensation area (see, e.g., Fig. 8 in work [9]) is determined by the mean value of $U\cos\alpha$. Here $0 \leqslant \alpha \leqslant \pi/2$ is a random angle between velocity at this streamline and radius-vector r at the point $r = r_1$, where the air enters the condensation area. From $U = 13\ \mathrm{m\,s^{-1}}$ [5] we have for dimensionless variables $u_1 = U/\pi u_c = 0.06$, $v_1 = 2U/\pi u_c = 0.12$. For $a = v_1 = 0.12$, we obtain the eye radius $r_0 = 0.074$ from Eq. (10). Taking cloud height $h = 1.2$ km and total size of tornado condensation area $r_1 = 8.5$ km, we have dimensionless value $h = 0.14$.

For these particular parameters the numerical solution of Eq. (7) obtained by using conditions (10) and (11) corresponds to $y_1' = 0.03574$ (see Fig. 1A). The account of stationary eye rotation is made in the same way as in work [2], when a certain part of tangential kinetic energy developed in the condensation area is spent on solid-body rotation and creation of the pressure gradient in the eye of radius r_0. This lowers tangential velocity in the transitional region $r_0 \leqslant r < r_e$ between the condensation area and the eye, where $r_e = 1.65 r_0$ [2]. The expressions for tangential velocity and pressure at $r < r_0$ coincide with Eqs. (23)–(25) in work [2]. The empirical points shown in Fig. 1B characterize the Mulhall tornado close to the time of peak intensity. They correspond to characteristic vertical velocity $w(r)$ at $z = 650$ m [5, Fig. 4b], mean radial velocity $u(r)$ at 150 m $< z <$ 850 m [5, Fig. 4b] and mean tangential velocity $v(r)$ at 50 m $< z <$ 950 m [5, Fig. 5a].

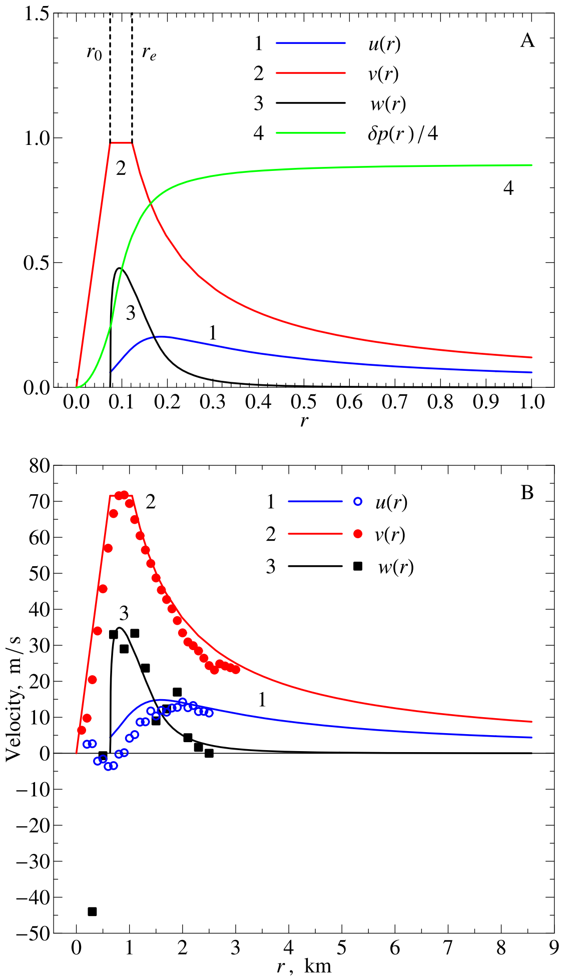

Fig. 1. (Color online.) A: Solution of Eq. (7) at $a = 0.12$ ($r_0 = 0.074$), $u_1 = 0.06$, $h = 0.14$ in dimensionless units (4), $\delta p(r) \equiv p(r) - p(0)$. B: Comparison with observations for the Mulhall tornado [5] at $u_c = 73\ \mathrm{m\,s^{-1}}$, $h = 1.2$ km ($r_1 = 8.5$ km). The negative vertical velocity (downdraft) within the tornado eye and the decrease of radial velocity near r_0 is related to non-stationarity of eye rotation not described by the Bernoulli integral (7), the latter pertaining to the converging ascending streamline.

It is seen from Fig. 1B that to the right side of the maximum the radial distribution of mean tangential velocity at $z \leqslant h$ conforms well to the assumption of conserved angular momentum (3). The choice of $h = 1.2$ km is supported by observation that in this layer the radial velocity $u(r)$ exceeds $u_1 = 4.4\ \mathrm{m\,s^{-1}}$ over considerable part of tornado circulation [5, Fig. 4b]. (It should be noted that two other tornadoes, for which the data are available [3,4], have a significantly lower inflow level, $h \leqslant 400$ m, than the Mulhall tornado. Mean tangential velocities do not follow the conserved angular momentum distribution. The decrease of angular momentum towards the center demands a more detailed consideration with additional parameters [2].)

Total pressure fall as shown in Fig. 1A is $3.6\Delta p = 108$ hPa. This is in agreement with the few available direct measurements of tornado surface pressure. In the Manchester tornado (South Dacota,

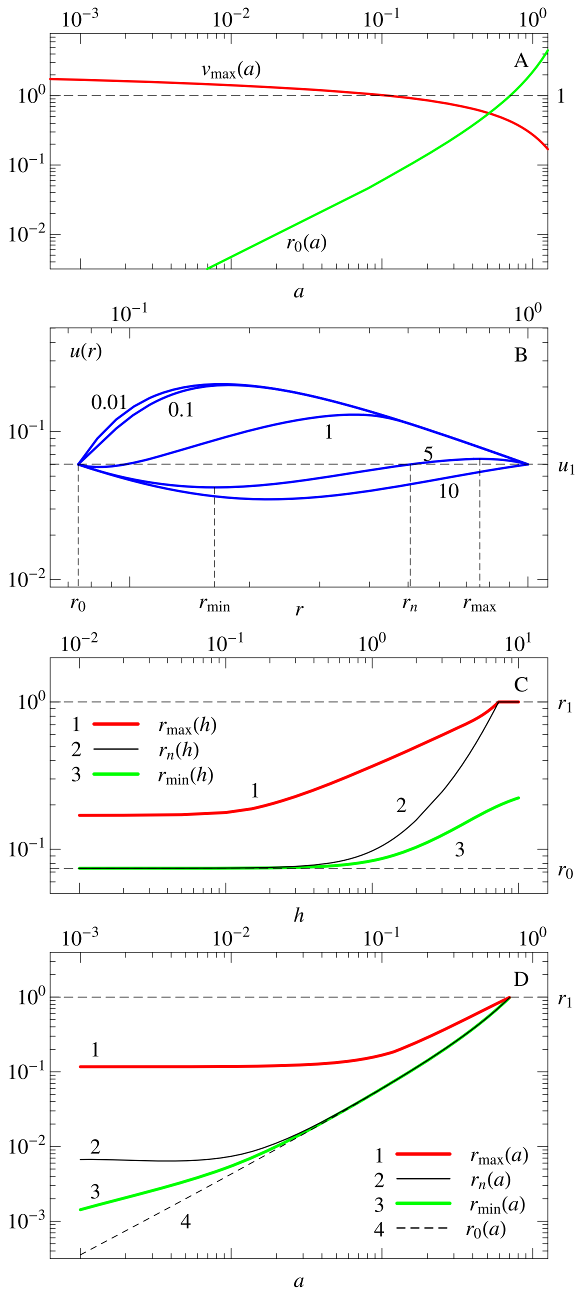

Fig. 2. (Color online.) A: Dependence of eye radius r_0 and maximal tangential velocity $v_{max} = a/r_e$, $r_e = 1.65r_0$ [2] on angular momentum a in dimensionless units (4). B: Radial velocity $u(r)$ obtained by solving Eq. (7) under conditions (10) and (11) at $a = 0.12$, $u_1 = 0.06$ for five values of h (shown near the curves). Points r_{min}, r_{max} and r_n shown for $h = 5$ correspond to radial velocity minimum, maximum and $u(r_n) = u_1$, respectively. C: Dependence of r_{min}, r_{max} and r_n on h at $a = 0.12$, $u_1 = 0.06$. D: Dependence of r_{min}, r_{max} and r_n on a at $h = 0.14$, $u_1 = 0.06$; $r_0(a)$ (10).

2003), which was of the same (F4) intensity as the Mulhall tornado, a pressure fall of 100 hPa was registered [5].

4. Conditions of vortex existence and the maximum possible velocities

With account of stationary eye rotation [2] the maximum wind velocity $v_{max} = a/r_e$ (and, correspondingly, the maximum kinetic energy) is achieved at $r_e = 1.65r_0$ (see Fig. 1A), where r_0 is a function of a given by Eq. (10), Fig. 2A. The Earth rotation does not determine angular momentum in tornado due to the small linear size of the vortex. The value of a is related to the translational velocity U. This velocity cannot be infinitely small: tornado exists at the expense of water vapor accumulated in the atmosphere and, hence, must move to sustain itself [10, pp. 227–229]. Maximum velocity attainable in the condensational vortex depends only weakly on angular momentum and grows rather slowly (logarithmically) with decreasing a (see Fig. 2A). For realistic $a \geqslant 10^{-3}$ ($v_1 \geqslant 0.1$ m s^{-1}), v_{max} does not exceed $1.7u_c \simeq 120$ m s^{-1}. This agrees well with the available estimates of maximum wind speeds in tornadoes [5].

The existence of vortex is related to a certain minimum value of radial velocity u, which describes the atmospheric inertia with respect to the development of condensational circulation. At $u < u_1$ condensation ceases. The condition $u_0 = u_1$ corresponds to the following relationships

$$u'(r_0) = \frac{u_0}{r_0}(y_0' - 1) = u_1\left(y_1' - \frac{1}{r_0}\right) \simeq -\frac{u_1}{r_0}, \tag{13}$$

$$u'(r_1) = u_0\left(\frac{y_0'}{r_0} - 1\right) = u_1(y_1' - 1). \tag{14}$$

This means that there is a minimum of $u(r)$ at $r = r_{min}$ within $r_0 < r_{min} < r_1$. The existence of condensation at $u(r) > u_1$ means that there is also a point $r = r_{max}$, $r_0 < r_{max} < r_1$, where $u(r)$ is maximum. It follows that there is a point $r = r_n$, $r_0 < r_{min} < r_n < r_{max} < r_1$, where $u(r_n) = u_1$. At $r < r_n$ there is no condensation and no condensational pressure potential to accelerate air. When $r_n \gg r_0$ the maximum vortex velocity a/r_e, is not reached: $v_{max} = a/r_n \ll a/r_e$. Therefore, tornado exists, if the following condition $\kappa \equiv (r_n - r_0)/r_0 \ll 1$ is fulfilled ($\kappa \simeq 10^{-3}$ for the vortex shown in Fig. 1).

Analysis of Eq. (7) shows that this condition is violated with increasing h, which, at a fixed height of the atmosphere, corresponds to diminishing linear size r_1 of the condensation area. In Fig. 2B, profiles of $u(r)$ are shown for h varying from 0.01 to 10. At $h \geqslant 7.3$ we have $y_1' \geqslant 1$, $u'(r_1) \geqslant 0$ and maximum of $u(r)$ at $r < r_1$ disappears. Decreasing a at fixed h also leads to increasing κ, Fig. 2D. It follows that the smaller the horizontal size of the condensation area, the larger the angular momentum that is needed for a vortex to arise. A given value of angular momentum sets the minimal horizontal size of the vortex. For the parameters shown in Fig. 1 the minimum possible vortex, where velocity $v \simeq u_c \simeq 70$ m s^{-1} can be observed, corresponds to $h \sim 1$ (see Fig. 2C). The minimal condensation area has then radius $r_1 \sim 1.2$ km and funnel (eye) radius of about 90 m. At small a and r_1 only ordinary squalls can develop.

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